

# Weighted Kolmogorov-Smirnov test: accounting for the tails

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Accurate Goodness of Fit tests for the extreme tails of empirical distributions is a very important issue, relevant in many contexts, including geophysics, insurance and finance. We have derived exact asymptotic results for a generalization of the Kolmogorov-Smirnov test, well suited to test these extreme tails. In passing, we have rederived and made more precise the result of [P. L. Krapivsky and S. Redner, *Am. J. Phys.* **64**(5):546, 1996] concerning the survival probability of a diffusive particle in an expanding cage.

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## I. INTRODUCTION AND MOTIVATION

The problem of testing whether a null-hypothesis theoretical probability distribution is compatible with the empirical probability distribution of a sample of observations is known as “Goodness-of-fit testing” and is ubiquitous in all fields of science and engineering. The best known theoretical result is due to Kolmogorov and Smirnov (KS) [2, 3], and has led to the eponymous statistical test. Several specific cases have been studied (and/or are still under scrutiny), including: univariate or multivariate samples [4–7], independent or dependent data [8], different choices of distance measures [9], investigation of different parts of the distribution domain [10, 11], etc.

This class of problems has a particular appeal for physicists since the works of Doob [12] and Khmaladze [13], who have shown how GoF testing is related to stochastic processes. Finding the law of a test often amounts to treating a Fokker-Planck problem, which in turn maps into a Schrödinger equation for a particle in a certain potential confined by walls.

The classical KS test suffers from an important flaw: the test is only weakly sensitive to the quality of the fit in the tails of the tested distribution, when it is often these tail events (corresponding to centennial floods, devastating earthquakes, financial crashes, etc.) that one is most concerned with. Here we focus on a GoF test for a univariate sample, with the Kolmogorov distance but equi-weighted quantiles, which is equally sensitive to all regions of the distribution. We unify two earlier attempts at finding asymptotic solutions, one by Anderson and Darling in 1952 [10] and a more recent, seemingly unrelated one that deals with “life and death of a particle in an expanding cage” by Krapivsky and Redner [1, 14]. We present here the exact asymptotic solution of the corresponding stochastic problem, and deduce from it the precise formulation of the GoF test, which is of a fundamentally different nature than the KS test.

## II. EMPIRICAL CUMULATIVE DISTRIBUTION AND ITS FLUCTUATIONS

Let  $\mathbf{X}$  be a latent random vector of  $N$  iid variables, with marginal cumulative distribution function (cdf)  $F$ . One realization of  $\mathbf{X}$  consists of a time series  $\{x_1, \dots, x_n, \dots, x_N\}$  that exhibits no persistence (see [8] when some non trivial dependence is present). For a given number  $x$  in the support of  $F$ , let  $\mathbf{Y}(x)$  be the random vector the components of which are the Bernoulli variables  $\mathbf{Y}_n(x) = \mathbb{1}_{\{\mathbf{X}_n \leq x\}}$ . The expected value and the covariance of  $\mathbf{Y}_n(x)$  are given by:

$$\begin{aligned} \mathbb{E}[\mathbf{Y}_n(x)] &= F(x), \\ \text{Cov}(\mathbf{Y}_n(x), \mathbf{Y}_m(x')) &= F(x)F(x')\delta_{nm} \end{aligned}$$

The centered sample mean of  $\mathbf{Y}(x)$  is:

$$\bar{\mathbf{Y}}(x) = \frac{1}{N} \sum_{n=1}^N \mathbf{Y}_n(x) - F(x) \quad (1)$$

which measures the difference between the empirically determined cdf at point  $x$  and its true value. It is therefore the quantity on which any statistics for Goodness-of-Fit testing is built. Denoting  $u = F(x)$  and  $v = F(x')$ , the covariance function of  $\bar{\mathbf{Y}}$  is easily shown to be:

$$\text{Cov}(\bar{\mathbf{Y}}(u), \bar{\mathbf{Y}}(v)) = \frac{1}{N} (\min(u, v) - uv)$$

(with a slight abuse of notation), where now and in the following

$$\bar{\mathbf{Y}}(u) = \frac{1}{N} \sum_{n=1}^N \mathbf{Y}_n(F^{-1}(u)) - u. \quad (1')$$

### Limit properties

One now defines the process  $y(u)$  as the limit of  $\sqrt{N}\bar{\mathbf{Y}}(u)$  when  $N \rightarrow \infty$ . For a given  $u$ , it represents

the difference between the empirically determined cdf of the (infinitely many)  $\mathbf{X}$ 's and the theoretical one, evaluated at the  $u$ -th quantile. According to the Central Limit Theorem, it is Gaussian and its covariance function is given by:

$$I(u, v) = \min(u, v) - uv \quad (2)$$

which characterizes the so-called Brownian bridge, i.e. a Brownian motion  $y(u)$  such that  $y(u=0) = y(u=1) = 0$ .

### Norms over processes and the Kolmogorov-Smirnov test

In order to measure a limit distance between distributions, a norm  $\|\cdot\|$  over the space of continuous bridges needs to be chosen. Interestingly, Eq. (2) does not depend explicitly on  $F$ , so that the law of  $\|y\|$  under any norm is distribution free.

Typical such norms are the norm-2 (or 'Cramer-von Mises' distance)

$$\|y\|_2 = \int_0^1 y(u)^2 du$$

as the bridge is always integrable, or the norm-sup

$$\|y\|_\infty = \sup_{u \in [0,1]} |y(u)|$$

as the bridge always reaches an extremal value (also called the Kolmogorov distance). Unfortunately, both these norms mechanically overweight the core values  $u \approx 1/2$  and disfavor the tails  $u \approx 0, 1$ : since the variance of  $y(u)$  is zero at both extremes and maximal in the central value, the major contribution to  $\|y\|$  indeed comes from the central region. In order to alleviate this effect — in particular when the GoF test is intended to investigate a specific region of the domain —, it is preferable to introduce additional weights and study  $\|y\sqrt{\psi}\|$  rather than  $\|y\|$  itself. Anderson and Darling show in Ref. [10] that the solution to the problem with the Cramer-von Mises norm and arbitrary weights  $\psi$  is obtained by spectral decomposition of the covariance kernel, and use of Mercer's theorem. In this note we will rather focus on the case  $\psi(u) = 1/V[y(u)]$ , which equi-weights all quantiles, and with the Kolmogorov distance, for which (to the best of our knowledge) no exact result has been reported in the literature.

### III. THE WEIGHTED BROWNIAN BRIDGE: LAW OF THE SUPREMUM

So again  $y(u)$  is a Brownian bridge, i.e. a centered Gaussian process on  $u \in [0, 1]$  with covariance function

$$\text{Cov}(y(u), y(u')) = \min(u, u') - uu'.$$

In particular,  $y(0) = y(1) = 0$  with probability equal to 1, no matter how distant  $F$  is from the sample cdf around the core values. In order to zoom on these tiny differences in the tails, we weight the Brownian bridge as follows: for given  $a \in ]0, 1[$  and  $b \in [a, 1[$ , we define

$$\tilde{y}(u) \equiv y(u)\sqrt{\psi(u; a, b)} \quad (3)$$

with

$$\psi(u; a, b) = \begin{cases} \frac{1}{u(1-u)}, & a \leq u \leq b \\ 0, & \text{otherwise.} \end{cases}$$

We will characterize the law of the supremum  $K(a, b) \equiv \sup_{u \in [0,1]} |\tilde{y}(u)|$ :

$$\begin{aligned} \mathcal{P}_<(k|a, b) &\equiv \mathbb{P}[K(a, b) \leq k] \\ &= \mathbb{P}[|\tilde{y}(u)| \leq k, \forall u \in [a, b]]. \end{aligned}$$

#### Diffusion in a cage with moving walls

Define the time change  $t = \frac{u}{1-u}$ . The variable  $W(t) = (1+t)y\left(\frac{t}{1+t}\right)$  is then a Brownian motion (Wiener process) on  $[\frac{a}{1-a}, \frac{b}{1-b}]$ , since one can check that:

$$\text{Cov}(W(t), W(t')) = \min(t, t').$$

$\mathcal{P}_<(k|a, b)$  can be now written as

$$\mathcal{P}_<(k|a, b) = \mathbb{P}\left[|W(t)| \leq k\sqrt{t}, \forall t \in [\frac{a}{1-a}, \frac{b}{1-b}]\right].$$

Remarks:

- The problem with initial time  $\frac{a}{1-a} = 0$  and horizon time  $\frac{b}{1-b} = T$  has been treated by Krapivsky and Redner in Ref. [1] as the survival probability  $S(T; k = \sqrt{\frac{A}{2D}})$  of a Brownian particle diffusing with constant  $D$  in a cage with walls expanding as  $\sqrt{At}$ . Their result is that for large  $T$ ,

$$S(T; k) \equiv \mathcal{P}_<(k|0, \frac{T}{1+T}) \propto T^{-\theta(k)}.$$

They obtain analytical expressions for  $\theta(k)$  in both asymptotic limits  $k \rightarrow 0$  and  $k \rightarrow \infty$ . We take here a slightly different route, suggested by Anderson and Darling in Ref. [10] but where the authors did not come to a conclusion. Our contributions are: (i) we treat the general case  $a > 0$  for *any*  $k$ ; (ii) we explicitly compute the  $k$ -dependence of both the exponent *and* the prefactor of the power-law decay; (iii) we provide the link with the theory of GoF tests and compute the pre-asymptotic distribution when  $]a, b[ \rightarrow ]0, 1[$  of the weighted Kolmogorov-Smirnov test statistics.

- Choosing a constant weight function  $\psi$  instead of the one above corresponds to the usual KS case and leads, after appropriate change of variable and time change, to a similar problem of a Brownian diffusion inside a box with walls moving at *constant* velocity. Since the walls now expand as  $Vt$  faster than the diffusive particle can move, the survival probability clearly decays to a positive value. The resulting survival probability turns out to be the usual Kolmogorov-Smirnov distribution.
- Other choices of  $\psi$  apparently result in much harder problems, see Ref. [10].

### An Ornstein-Uhlenbeck process with fixed walls

Introducing now the new time change  $\tau = \log \sqrt{\frac{1-a}{a}} t$ , the variable  $Z(\tau) = W(t)/\sqrt{t}$  is a stationary Ornstein-Uhlenbeck process on  $[0, T]$  where

$$T = \log \sqrt{\frac{b(1-a)}{a(1-b)}}, \quad (4)$$

and

$$\text{Cov}(Z(\tau), Z(\tau')) = e^{-|\tau - \tau'|}.$$

Its dynamics is described by the Stochastic Differential Equation

$$dZ(T) = -Z(T)dT + \sqrt{2}dB(T) \quad (5)$$

with  $B(T)$  an independent Wiener process. The initial condition for  $T = 0$  (corresponding to  $b = a$ ) is  $Z(0) = y(a)/\sqrt{V[y(a)]}$ , a random Gaussian variable of zero mean and unit variance. The distribution  $\mathcal{P}_<(k|a, b)$  can now be understood as the unconditional survival probability of a mean-reverting particle in a cage with fixed absorbing walls:

$$\begin{aligned} \mathcal{P}_<(k|T) &= \mathbb{P}[-k \leq Z(\tau) \leq k, \forall \tau \in [0, T]] \\ &= \int_{-k}^k f_T(z; k) dz \end{aligned}$$

where

$$f_T(z; k) dz = \mathbb{P}[Z(T) \in [z, z + dz] | \{Z(\tau)\}_{\tau < T}]$$

is the density probability of the particle being at  $z$  at time  $T$ , when walls are in  $\pm k$ . Its dependence on  $k$ , although not explicit on the right hand side, is due to the boundary condition associated with the absorbing walls (it will be dropped in the following for the sake of readability) [19].

The Fokker-Planck equation governing the evolution of the density  $f_T(z)$  reads

$$\partial_\tau f_\tau(z) = \partial_z [z f_\tau(z)] + \partial_z^2 [f_\tau(z)], \quad 0 < \tau \leq T.$$

Calling  $\mathcal{H}_{FP}$  the second order differential operator  $-\left[\mathbf{1} + z\partial_z + \partial_z^2\right]$ , the full problem thus amounts to finding the general solution of

$$\begin{cases} -\partial_\tau f_\tau(z) = \mathcal{H}_{FP}(z) f_\tau(z) \\ f_\tau(\pm k) = 0, \forall \tau \in [0, T] \end{cases}.$$

We have explicitly introduced a minus sign since we expect that the density decays with time in an absorption problem. Because of the term  $z\partial_z$ ,  $\mathcal{H}_{FP}$  is not hermitian and thus cannot be diagonalized. However, as is well known, one can define  $f_\tau(z) = e^{-\frac{z^2}{4}} \phi_\tau(z)$  and the Fokker-Planck equation becomes

$$\begin{cases} -\partial_\tau \phi_\tau(z) = \left[-\partial_z^2 + \frac{1}{4}z^2 - \frac{1}{2}\mathbf{1}\right] \phi_\tau(z) \\ \phi_\tau(\pm k) = 0, \forall \tau \in [0, T] \end{cases}$$

and its Green function, i.e. the (separable) solution *conditionally on the initial position*  $(z_i, T_i)$ , is the superposition of all modes

$$G_\phi(z, T | z_i, T_i) = \sum_\nu e^{-\theta_\nu(T-T_i)} \hat{\varphi}_\nu(z) \hat{\varphi}_\nu(z_i),$$

where  $\hat{\varphi}_\nu$  are the normalized solutions of the stationary Schrödinger equation:

$$\begin{cases} \left[-\partial_z^2 + \frac{1}{4}z^2\right] \varphi_\nu(z) = (\theta_\nu + \frac{1}{2}) \varphi_\nu(z) \\ \varphi_\nu(\pm k) = 0 \end{cases}$$

each decaying with its own energy  $\theta_\nu$ , where  $\nu$  labels the different solutions with increasing eigenvalues, and the set of eigenfunctions  $\{\hat{\varphi}_\nu\}$  defines an orthonormal basis of the Hilbert space on which  $\mathcal{H}_S(z) = \left[-\partial_z^2 + \frac{1}{4}z^2\right]$  acts. In particular,

$$\sum_\nu \hat{\varphi}_\nu(z) \hat{\varphi}_\nu(z') = \delta(z - z'), \quad (6)$$

so that indeed  $G(z, T_i | z_i, T_i) = \delta(z - z_i)$ , and the general solution writes

$$\begin{aligned} f_T(z_T; k) &= \int_{-k}^k \frac{e^{-\frac{z_T^2}{4}}}{e^{-\frac{z_i^2}{4}}} G_\phi(z_T, T | z_i, T_i) f_0(z_i) dz_i \\ &= \int_{-k}^k e^{\frac{z_i^2 - z_T^2}{4}} G_\phi(z_T, T | z_i, T_i) f_0(z_i) dz_i \end{aligned}$$

where  $T_i = 0$ , which corresponds to the case  $b = a$  in Eq. (3), and  $f_0$  is the distribution of the initial value  $z_i$  which is here, as noted above, Gaussian with unit variance.

$\mathcal{H}_S$  figures out an harmonic oscillator of mass  $\frac{1}{2}$  and frequency  $\omega = \frac{1}{\sqrt{2}}$  within an infinitely deep well of width  $2k$ : its eigenfunctions are parabolic cylinder functions [15, 16]

$$\begin{aligned} y_+(\theta; z) &= e^{-\frac{z^2}{4}} {}_1F_1\left(-\frac{\theta}{2}, \frac{1}{2}, \frac{z^2}{2}\right) \\ y_-(\theta; z) &= z e^{-\frac{z^2}{4}} {}_1F_1\left(\frac{1-\theta}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \end{aligned}$$

properly normalized. The only acceptable solutions for a given problem are the linear combinations of  $y_+$  and  $y_-$  which satisfy orthonormality (6) and the boundary conditions: for periodic boundary conditions, only the integer values of  $\theta$  would be allowed, whereas with our Dirichlet boundaries  $|\hat{\varphi}_\nu(k)| = -|\hat{\varphi}_\nu(-k)| = 0$ , real non-integer eigenvalues  $\theta$  are allowed [20]. For instance, the fundamental level  $\nu = 0$  is expected to be the symmetric solution  $\hat{\varphi}_0(z) \propto y_+(\theta_0; z)$  with  $\theta_0$  the smallest possible value compatible with the boundary condition. The boundary condition in fact provides the implicit equation for  $\theta_0$ :

$$\theta_0(k) = \inf_{\theta > 0} \{ \theta : y_+(\theta; k) = 0 \}. \quad (7)$$

In what follows, it will be more convenient to make the  $k$ -dependence explicit, and a hat will denote the solution with the normalization relevant to our problem, namely  $\hat{\varphi}_0(z; k) = y_+(\theta_0(k); z) / \|y_+\|_k$ , with the norm

$$\|y_+\|_k^2 \equiv \int_{-k}^k y_+(\theta_0(k); z)^2 dz$$

so that  $\int_{-k}^k \hat{\varphi}_\nu(z; k)^2 dz = 1$ .

### Asymptotic survival rate

Denoting by  $\Delta_\nu(k) \equiv (\theta_\nu(k) - \theta_0(k))$  the gap between the excited levels and the fundamental, the higher energy modes  $\hat{\varphi}_\nu$  cease to contribute to the Green function when  $\Delta_\nu T \gg 1$ , and their contribution to the above sum die out exponentially as  $T$  grows. Eventually, only the lowest energy mode  $\theta_0(k)$  remains, and the solution tends to

$$f_T(z; k) = A(k) e^{-\frac{z^2}{4}} \hat{\varphi}_0(z; k) e^{-\theta_0(k)T}$$

when  $T \gg (\Delta_1)^{-1}$ , with

$$A(k) = \int_{-k}^k e^{\frac{z^2}{4}} \hat{\varphi}_0(z; k) f_0(z) dz. \quad (8)$$

Let us come back to the initial problem of the weighted Brownian bridge reaching its extremal value in  $[a, b]$ . If we are interested in the limit case where  $a$  is arbitrarily close to 0 and  $b$  close to 1, then  $T \rightarrow \infty$  and the solution is thus given by:

$$\begin{aligned} \mathcal{P}_<(k|T) &= A(k) e^{-\theta_0(k)T} \int_{-k}^k e^{-\frac{z^2}{4}} \hat{\varphi}_0(z; k) dz \\ &= \tilde{A}(k) e^{-\theta_0(k)T}, \end{aligned}$$

with  $\tilde{A}(k) \equiv \sqrt{2\pi} A(k)^2$ .

We now compute explicitly the asymptotic behaviour of both  $\theta_0(k)$  and  $\tilde{A}(k)$ :

**$k \rightarrow \infty$**  As  $k$  goes to infinity, the absorption rate  $\theta_0(k)$  is expected to converge toward 0: intuitively, an infinitely far barrier will not absorb anything. At the same time,  $\mathcal{P}_<(k|T)$  must tend to 1 in that limit. So  $\tilde{A}(k)$  necessarily tends to one. Indeed,

$$\begin{aligned} \theta_0(k) &\xrightarrow{k \rightarrow \infty} \sqrt{\frac{2}{\pi}} k e^{-\frac{k^2}{2}} \rightarrow 0 \\ \tilde{A}(k) &\xrightarrow{k \rightarrow \infty} \left( \int_{-\infty}^{\infty} \hat{\varphi}_0(z; \infty)^2 dz \right)^2 = 1. \end{aligned} \quad (9)$$

In principle, we see from Eq. (8) that corrections to the later arise both (and jointly) from the functional relative difference of the solution  $\epsilon(z; k) = y_+(\theta_0(k); z) / y_+(0; z) - 1$ , and from the finite integration limits ( $\pm k$  instead of  $\pm \infty$ ). However, it turns out that the correction of the first kind is of second order in  $\epsilon$  [21]. The correction to  $A(k)$  is thus dominated by the finite integration limits ( $\pm k$ ), so that pre-asymptotically:

$$\tilde{A}(k \rightarrow \infty) \approx \text{erf} \left( \frac{k}{\sqrt{2}} \right). \quad (10)$$

**$k \rightarrow 0$**  For small  $k$ , the system behaves like a free particle in a sharp and infinitely deep well, since the quadratic potential is almost flat around 0. The fundamental mode becomes then

$$\hat{\varphi}_0(z; k \rightarrow 0) = \frac{1}{\sqrt{k}} \cos \left( \frac{\pi z}{2k} \right)$$

and consequently

$$\theta_0(k) \xrightarrow{k \rightarrow 0} \frac{\pi^2}{4k^2} - \frac{1}{2} \quad (11)$$

$$\begin{aligned} \tilde{A}(k) &\xrightarrow{k \rightarrow 0} \left( \int_{-k}^k \frac{e^{-\frac{z^2}{4}}}{(2\pi)^{\frac{1}{4}} \sqrt{k}} \cos \left( \frac{\pi z}{2k} \right) dz \right)^2 \\ &\approx \frac{1}{\sqrt{2\pi} k} \left( \frac{4k}{\pi} \right)^2 = \frac{16}{\pi^2 \sqrt{2\pi}} k. \end{aligned} \quad (12)$$

We show in Fig. 1 the functions  $\theta_0(k)$  and  $\tilde{A}(k)$  computed numerically from the exact solution, together with their asymptotic analytic expressions. In intermediate values of  $k$  (roughly between 0.5 and 3) these asymptotic expressions fail to reproduce the exact solution.

### Higher modes and validity of the asymptotic ( $N \gg 1$ ) solution.

Higher modes  $\nu > 0$  with energy gaps  $\Delta_\nu \lesssim 1/T$  must in principle be kept in the pre-asymptotic computation. This however is irrelevant in practice since the gap  $\theta_1 - \theta_0$  is never small. Indeed,  $\hat{\varphi}_1(z; k)$  is proportional to the asymmetric solution  $y_-(\theta_1(k); z)$  and its energy

$$\theta_1(k) = \inf_{\theta > \theta_0(k)} \{ \theta : y_-(\theta; k) = 0 \}$$

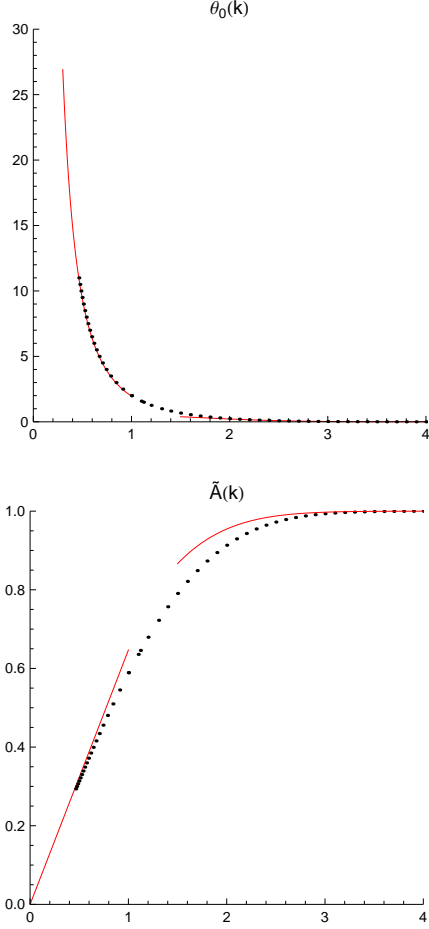


FIG. 1: **Top:** Dependence of the exponent  $\theta_0$  on  $k$ ; similar to Fig. 2 in Ref. [1] — see in particular Eqs. (9b) and (12) there. **Bottom:** Dependence of the prefactor  $\tilde{A}$  on  $k$ . The red plain lines illustrate the analytical behavior in the limiting cases  $k \rightarrow 0$  and  $k \rightarrow \infty$ .

is found numerically to be very close to  $1 + 4\theta_0(k)$ . In particular,  $\Delta_1 > 1$  (as we illustrate in Fig. 2) and thus  $T\Delta_1 \gg 1$  will always be satisfied in cases of interest.

#### IV. BACK TO GOF TESTING AND CONCLUSION

Let us now come back to GoF testing. In the case of a constant weight, corresponding to the classical KS test, the probability  $\mathcal{P}_<(k|a=0, b=0)$  is well defined and has the well known KS form [2]:

$$\mathcal{P}_<(k|a=0, b=0) = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-2n^2 k^2},$$

which, as expected, grows from 0 to 1 as  $k$  increases. The value  $k^*$  such that this probability is 95% is  $k^* \approx 1.358$  [3]. This can be interpreted as follows: if, for a data

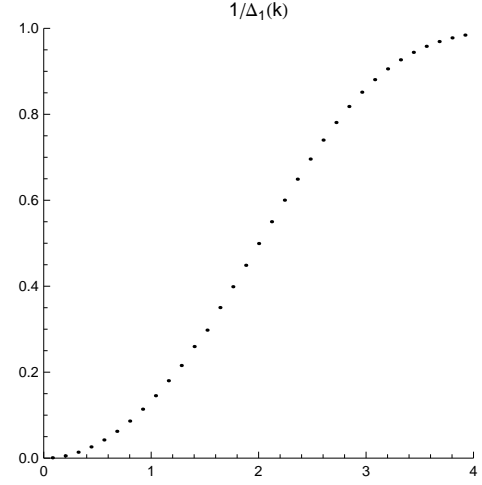


FIG. 2:  $1/\Delta_1(k)$  saturates to 1, so that the condition  $N \gg \exp(1/\Delta_1(k))$  is virtually always satisfied.

set of size  $N$ , the maximum value of  $\bar{\mathbf{Y}}(u)$  is larger than  $\approx 1.358/\sqrt{N}$ , then the hypothesis that the proposed distribution is a “good fit” can be rejected with 95% confidence.

In order to convert the above calculations into a meaningful test, one must specify values of  $a$  and  $b$ . The natural choice is  $a = 1/N$  and  $b = 1 - a$ , corresponding to the quantiles of the min and the max of the sample series. Indeed,  $a = F(\min z) \approx \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{\{z_n \leq \min z\}} = \frac{1}{N}$ , and similarly for  $b$ . Correspondingly, the relevant value of  $T$  is given, according to Eq. (4) above, by

$$T = \log \sqrt{\frac{b(1-a)}{a(1-b)}} \approx \log N, \quad N \gg 1.$$

This leads to our central result for the cdf of the weighted maximal Kolmogorov distance  $K(\frac{1}{N+1}, \frac{N}{N+1})$  under the hypothesis that the tested and the true distributions coincide:

$$\boxed{S(N; k) = \mathcal{P}_<(k|\log N) = \tilde{A}(k)N^{-\theta_0(k)}} \quad (13)$$

which is valid whenever  $N \gg 1$  since, as we discussed above, the energy gap  $\Delta_1$  is greater than unity.

The final cumulative distribution function (the test law) is depicted in Fig. 3 for different values of the sample size  $N$ . Contrarily to the standard KS case, this distribution *still depends on*  $N$ . In particular, the threshold value  $k^*$  corresponding to a 95% confidence level increases with  $N$ . Since for large  $N$ ,  $k^* \gg 1$  one can use the asymptotic expansion above which soon becomes quite accurate, as shown in Fig. 3. This leads to:

$$\theta_0(k^*) \approx -\frac{\ln 0.95}{\ln N} \approx \sqrt{\frac{2}{\pi}} k^* e^{-\frac{k^{*2}}{2}},$$

which gives  $k^* \approx 3.439, 3.529, 3.597, 3.651$  for, respectively,  $N = 10^3, 10^4, 10^5, 10^6$ . For exponentially large

$N$  and to logarithmic accuracy one has:  $k^* \sim \sqrt{2 \ln \ln N}$ . This variation is very slow, but one sees that as a matter of principle, the “acceptable” maximal value of the weighted distance is much larger (for large  $N$ ) than in the KS case.

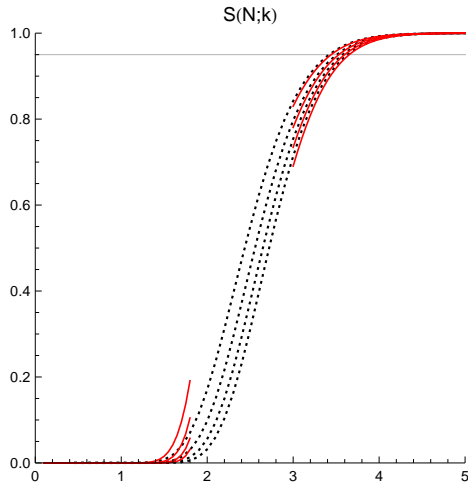


FIG. 3: Dependence of  $S(N; k)$  on  $k$  for  $N = 10^3, 10^4, 10^5, 10^6$  (from left to right). As  $N$  grows toward infinity, the curve is shifted to the right, and eventually  $S(\infty; k)$  is zero for any  $k$ . The red plain lines illustrate the analytical behavior in the limiting cases  $k \rightarrow 0$  and  $k \rightarrow \infty$ . The horizontal grey line corresponds to a 95% confidence level.

In conclusion, we believe that accurate GoF tests for the extreme tails of empirical distributions is a very important issue, relevant in many contexts. We have derived exact asymptotic results for a generalization of the Kolmogorov-Smirnov test, well suited to test these extreme tails. Our final results are summarized in Fig. 3. In passing, we have rederived and made more precise the result of Krapivsky and Redner [1] concerning the survival probability of a diffusive particle in an expanding cage. It would be interesting to exhibit other choices of weight functions that lead to soluble survival probabilities. It would also be interesting to extend the present results to multivariate distributions, and to dependent observations, along the lines of Ref. [8].

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  - [19] In particular,  $\mathcal{P}_<(k|0) = \text{erf}\left(\frac{k}{\sqrt{2}}\right)$ .
  - [20] A similar problem with a *one-sided* barrier leads to a continuous spectrum; this case has been studied originally in Ref. [15] and more recently in Ref. [17] (it is shown that there exists a quasi-stationary distribution for any  $\theta$ ) and generalized in Ref. [18].
  - [21] From Eq. (8) we have, when  $k \rightarrow \infty$ ,

$$A(k) = (2\pi)^{-1/2} \frac{\int_{-k}^k e^{-z^2/2} (1 + \epsilon(z; k)) dz}{\sqrt{\int_{-k}^k e^{-z^2/2} (1 + \epsilon(z; k))^2 dz}}.$$

The result follows by keeping only the dominant terms in the expansion in powers of  $\epsilon(z; k)$ . A similar computation for the asymptotic analysis by expanding the wave function in  $\theta$  was performed in Ref. [1]. Alternatively, algebraic arguments allow to understand that, to first order in the energy correction  $\theta_0(k) - \theta_0(\infty)$ , the perturbation of the wave function is orthogonal to  $\hat{\varphi}_0(z; \infty)$ .